Lateral-torsional buckling of orthotropic rectangular section beams

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Summary
The elastic lateral-torsional buckling of rectangular section beams made of wood under various loading conditions are investigated in this paper. Three dimensional finite element models considering the orthotropic behavior of wood are utilized to predict the elastic buckling load. The effects of depth/width ratios of the cross section, wood species, and cross grain of the beams on the predicted buckling loads are investigated. For all the cases considered, the critical loads predicted using finite element models assuming wood as orthotropic material are much smaller than those predicted assuming wood as isotropic material. The predicted critical loads of beams with cross grain of 45° are approximately a third of the predicted critical loads of beams without any cross grain. The finite element results agree very well with the beam tests on several species of the hardwood category.

1. Introduction
Lateral-torsional buckling (LTB) is a limit state where beam deformation includes in-plane deformation, out-of-plane deformation, and twisting [1]. Consider, for example, a simply supported rectangular beam loaded by uniform moment $M$ as seen in Figure 1. The beam is laterally supported at both ends so that the cross sections at both ends cannot rotate but free to warp. Assuming that the material is isotropic and linearly elastic, the critical moment, $M_{0cr}$, that causes lateral-torsional buckling to occur is

$$M_{cr} = \frac{\pi}{L_u} \sqrt{EI_y GJ}$$

Fig. 1 Lateral Torsional Buckling (LTB) of a rectangular section beam.

(1)

where $E$ = modulus of elasticity, $I_y$ = moment of inertia with respect to the weak (y) axis, $G$ = shear modulus, and $J$ = torsional constant, and $L_u$ is the unbraced length of the beam. For rectangular sections with depth $d$ and width $b$,

$$I_y = \frac{1}{12} b^3 d$$

(2)

$$J = k_1 b^3 d$$

(3)

where $k_1$ is a cross section constant and can be obtained from Table 1 [2].

Note that warping, a deformation that occurs in a thin walled open section, is neglected in Eq. (1). The derivation of Eq. (1) can be found in literature [3].

For nonuniform moment diagram, the critical moment can be obtained by multiplying the critical moment for uniform moment, Eq. (1), by an LTB modification factor for nonuniform moment $C_b$.
For a simply supported beam laterally supported at its ends under uniform load and, \( C_b = 1.14 \) and if the beam is loaded by concentrated load at midspan, \( C_b = 1.32 \) [5]. Therefore, the critical uniform load for a simply supported beam laterally supported at its ends is

\[
w_{cr} = \frac{8}{L_u^2} (1.14) \frac{\pi}{L_u} \sqrt{EI_y GJ}
\]

and the critical concentrated load at midspan for a simply supported beam laterally supported at its ends is

\[
P_{cr} = \frac{4}{L_u} (1.32) \frac{\pi}{L_u} \sqrt{EI_y GJ}
\]

It should be noted that Eq. (1), (4), and (5) were derived by assuming that the beam material is isotropic. In this paper, the applicability of these equations for orthotropic material is investigated by using Finite Element Method (FEM).

### Table 1 Constant \( k_1 \) required for computing torsional constant \( J \) of a rectangular section [2]

<table>
<thead>
<tr>
<th>d/b</th>
<th>( k_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.229</td>
</tr>
<tr>
<td>3</td>
<td>0.263</td>
</tr>
<tr>
<td>4</td>
<td>0.281</td>
</tr>
<tr>
<td>5</td>
<td>0.291</td>
</tr>
</tbody>
</table>

2. **Finite Element Method**

If wood is assumed as an orthotropic and elastic material with three mutually perpendicular material principal axes (longitudinal, radial, and tangential), then the relationship between strain components and stress components can be expressed as [6]

\[
\begin{bmatrix}
\varepsilon_L \\
\varepsilon_R \\
\varepsilon_T \\
\gamma_{LR} \\
\gamma_{LT} \\
\gamma_{RT}
\end{bmatrix} =
\begin{bmatrix}
1/E_L & -\mu_{LR}/E_R & -\mu_{LT}/E_T & 0 & 0 & 0 \\
-\mu_{LR}/E_L & 1/E_R & -\mu_{LT}/E_T & 0 & 0 & 0 \\
-\mu_{LT}/E_L & -\mu_{LR}/E_R & 1/E_T & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{LR} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{LT} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{RT}
\end{bmatrix}
\begin{bmatrix}
\sigma_L \\
\sigma_R \\
\sigma_T \\
\tau_{LR} \\
\tau_{LT} \\
\tau_{RT}
\end{bmatrix}
\]

where \( E_i \) are moduli of elasticity, \( \mu_{ij} \) are Poisson’s ratios, and \( G_{ij} \) are shear moduli, where \( i = L, R, \) and \( T. \) For an isotropic material, the relationship reduces to

\[
\begin{bmatrix}
\varepsilon_L \\
\varepsilon_R \\
\varepsilon_T \\
\gamma_{LR} \\
\gamma_{LT} \\
\gamma_{RT}
\end{bmatrix} =
\begin{bmatrix}
1/E & -\mu/E & -\mu/E & 0 & 0 & 0 \\
-\mu/E & 1/E & -\mu/E & 0 & 0 & 0 \\
-\mu/E & -\mu/E & 1/E & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G
\end{bmatrix}
\begin{bmatrix}
\sigma_L \\
\sigma_R \\
\sigma_T \\
\tau_{LR} \\
\tau_{LT} \\
\tau_{RT}
\end{bmatrix}
\]

where \( G = E/(2(1+\mu)). \)

In the subsequent analyses using a finite element program (SAP2000 Advanced 11.0.0 ©), the beam is modelled by using solid (3-D) elements. Boundary conditions of the finite element mesh are the same as the boundary conditions used in the derivation of Eq.(1), i.e. no out-of-plane deformation.
at both beam ends but both ends can rotate in the plane of the beam. By defining buckling analysis in the program, the critical load along with the buckling modes can be obtained. The first mode is the one that is associated with the critical (lowest) load. As seen in Fig. 2, the buckling mode shows in-plane and out-of-plane deformations indicating that it is LTB mode.

Eq. (6) and (7) are incorporated in the program for orthotropic and isotropic materials, respectively. The hypothetical beam is assumed to be made of walnut. The static bending modulus of elasticity $E_{sb}$ of the material is 11600 MPa [7]. The longitudinal modulus of elasticity $E_L$ can be taken as ten percent higher than the static bending modulus of elasticity [7], i.e. $E_L = 1.1 \times 11600$ MPa = 12760 MPa. The other elastic properties are taken from the literature [7], namely $E_R = 1353$ MPa, $E_T = 715$ MPa, $\mu_{LR} = 0.495$, $\mu_{LT} = 0.632$, $\mu_{RT} = 0.718$, $G_{LR} = 1085$ MPa, $G_{LT} = 0.791$, and $G_{RT} = 268$ MPa. The same beam is also analyzed assuming isotropic behaviour by using the modulus of elasticity $E = 12760$ MPa and Poisson’s ratio $\mu = 0.0525$.

The FEM is used to analyze isotropic and orthotropic beams under three loading conditions, namely uniform moment, distributed load, and concentrated load.

### 2.1 Uniform Moment

![Fig. 3 A simply supported beam under uniform bending moment diagram. Both ends of the beam are laterally supported.](image)

To investigate if FEM is appropriate in predicting critical load, consider a simply supported beam under uniform moment as seen in Fig. 3. The beam length $L = L_u = 800$ mm. The cross section is rectangular with the width $b = 40$ mm and depth $d = 80$ mm. Assuming the beam as isotropic material, it can be seen in Table 2 that FEM results in a critical moment $M_{cr} = 24.46$ kNm which is only 0.12% higher than the critical moment computed using Eq. (1). This result shows that FEM can be used in buckling analysis with a very high accuracy.

To investigate if the isotropic equation (Eq. 1) can be used in predicting the critical load of an orthotropic material, the same beam is further analyzed using FEM assuming orthotropic material. The beam plane is L-T plane. The analysis is repeated for several angles of cross grain, namely $0^\circ$, $15^\circ$, $30^\circ$, and $45^\circ$. Cross grain is defined as the angle between the beam longitudinal axis and the material L axis. As seen in Table 2 and Fig. 4, the critical moment for orthotropic material is much lower than the critical moment computed using Eq. (1). The difference is higher for greater cross grain angle. These results show that it is unsafe to use the isotropic equation (Eq. 1) for orthotropic material such as wood.

### Table 2 Critical moment $M_{cr}$ (kNm) computed using isotropic equation (Eq. 1), FEM assuming isotropic material, and FEM assuming orthotropic material.

<table>
<thead>
<tr>
<th>Method</th>
<th>$M_{cr}$ (kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic Equation (Eq. 1)</td>
<td>24.43</td>
</tr>
<tr>
<td>FEM, isotropic</td>
<td>24.46</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain $0^\circ$</td>
<td>9.07</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain $15^\circ$</td>
<td>7.51</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain $30^\circ$</td>
<td>4.88</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain $45^\circ$</td>
<td>3.17</td>
</tr>
</tbody>
</table>
Fig. 4 The effect of cross grain on the critical end moments $M_{cr}$ for a simply supported beam under uniform bending moment diagram.

### 2.2 Distributed Load

![Distributed Load](image)

Fig. 5 A simply supported beam under uniform load. Both ends of the beam are laterally supported.

<table>
<thead>
<tr>
<th>Method</th>
<th>$w_{cr}$ (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic Equation (Eq. 4)</td>
<td>348.09</td>
</tr>
<tr>
<td>FEM, isotropic</td>
<td>328.80</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain 0°</td>
<td>109.30</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain 15°</td>
<td>90.88</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain 30°</td>
<td>60.84</td>
</tr>
<tr>
<td>FEM, orthotropic, cross grain 45°</td>
<td>40.97</td>
</tr>
</tbody>
</table>

Table 3 Critical load $w_{cr}$ (kN/m) computed using isotropic equation (Eq. 4), FEM assuming isotropic material, and FEM assuming orthotropic material.

The same beam as described before is now loaded by uniformly distributed load $w$ as seen in Fig. 5. As before, the beam is analyzed assuming as isotropic material and orthotropic material with several angles of cross grain. As seen in Table 3 and Fig. 6, the critical load $w_{cr}$ obtained from FEM is only 5.54% lower than the critical load computed using Eq. 4. This result shows that, once again, FEM can be used in buckling analysis with a very high accuracy.

If the material is orthotropic, FEM shows that the critical load $w_{cr}$ is much lower than the critical load for isotropic material. For an orthotropic beam with cross grain of 45°, the critical load is only approximately 12% of the critical load of isotropic beam. The conclusion for this loading case is the same as in the case of uniform moment, i.e. it is unsafe to use the isotropic equation (Eq. 1) for orthotropic material such as wood.
Fig. 6 The effect of cross grain on the critical load $w_{cr}$ for a simply supported beam under uniformly distributed load.

2.3 Concentrated Load

To investigate the effect of cross section aspect ratio, $d/b$, on the critical buckling load, a hypothetical simply supported walnut beam under a concentrated load $P$ as seen in Fig. 7 is analyzed using FEM. The material properties are the same as described before. The beam length and width are still the same as before, i.e. $L = 800$ mm and $b = 40$ mm. The depth of the cross section varies to give $d/b$ ratios of 2, 3, 4, and 5.

As seen in Table 4 and Fig. 8 the difference between FEM results for isotropic material and Eq. 5 are quite small. The largest difference is 17.98% (for $d/b = 5$).

As in the loading cases analyzed before, the critical load obtained using FEM for orthotropic material is much smaller than that for isotropic material. The difference between isotropic and orthotropic critical loads is larger as the ratio $d/b$ increases. This is due to the fact that as $d/b$ increases, the beam behaves more like a deep beam and the anisotropic nature of the material becomes more pronounced.

Table 4 Comparison of critical loads $P_{cr}$ (kN) computed using isotropic equation (Eq. 5), FEM assuming isotropic material, and FEM assuming orthotropic material.

<table>
<thead>
<tr>
<th>$d/b$</th>
<th>Isotropic</th>
<th>FEM, isotropic</th>
<th>FEM, orthotropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>161.72</td>
<td>151.92</td>
<td>47.47</td>
</tr>
<tr>
<td>3</td>
<td>259.17</td>
<td>235.55</td>
<td>68.64</td>
</tr>
<tr>
<td>4</td>
<td>357.18</td>
<td>313.35</td>
<td>85.49</td>
</tr>
<tr>
<td>5</td>
<td>470.46</td>
<td>385.88</td>
<td>98.12</td>
</tr>
</tbody>
</table>
Fig. 8 Critical load $P_{cr}$ (kN) for various cross section aspect ratio $d/b$. The width $b$ is kept constant while depth $d$ varies.

3. Static Bending Tests

To verify the results of FEM, static bending tests on simply supported beams were performed. The experimental setup is shown in Fig. 9. As seen in the figure, both ends of the beam are laterally supported. The loading given by the Universal Testing Machine (UTM) was displacement controlled with a displacement rate of 2 mm/minute. A dial gauge was attached at midspan to monitor the lateral deformation. Three species in hardwood category grown in Indonesia were tested, namely albasia (albizia falcata, batai), meranti = shorea spec. div.), and bangkirai (hopea spec. div.). For each species, three specimens were tested. The specific gravity of each species were 0.90, 0.45, and 0.38 for bangkirai, meranti, and albasia, respectively. The moisture content of all specimens were approximately 15%. The dimensions of all specimens were as follows: total length $L_{total} = 1260$ mm, beam span $L = 1160$ mm, width $b = 20$ mm, and depth $d = 100$ mm.

By using the computer attached to the UTM, the load versus midspan deflection for each test was recorded. The typical load - midspan deflection curve is shown in Fig. 10. The static bending modulus of elasticity $E_{sb}$ can be computed using the elementary equation

$$E_{sb} = \frac{PL^3}{48\Delta I_s}$$

where load $P$ and midspan deflection $\Delta$ are the values during the linearly elastic stage of the curve. Static bending moduli of elasticity for each species computed using Eq. (8) are 12233 MPa, 5026 MPa, and 3091 MPa, for bangkirai, meranti, and albasia, respectively.
Fig. 10 Typical load (N) – deflection (mm) curve obtained from static bending test.

The critical load was taken as the load at which the beam started to deflect laterally as indicated by the dial gauge. The results for each species are shown in Table 5. As expected, the critical load $P_{cr}$ for the species with the highest specific gravity is the largest. These experimental results are compared with the FEM results.

<table>
<thead>
<tr>
<th>Species</th>
<th>FEM, orthotropic</th>
<th>Static bending test</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bangkirai</td>
<td>2529.9</td>
<td>3933.0</td>
<td>-35.7%</td>
</tr>
<tr>
<td>Meranti</td>
<td>2160.0</td>
<td>1833.0</td>
<td>17.8%</td>
</tr>
<tr>
<td>Albasia</td>
<td>1327.1</td>
<td>1317.0</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

Table 5 Comparison of critical loads $P_{cr}$ (N) obtained from FEM and static bending test.

To perform buckling analysis using FEM, material properties for each species are needed. As before, longitudinal modulus of elasticity $E_L = 1.1E_{sb}$. Since there is no data of all other orthotropic elastic properties for the three species tested, the properties were taken as shown in Table 6. The critical buckling loads $P_{cr}$ obtained from FEM are shown in Table 5. It is interesting to note that although the elastic properties in FEM are approximation, the difference between experimental result and FEM result is relatively small, especially for Albasia.

Table 6 Elastic properties used in finite element analyses of three species, namely bangkirai, meranti, and albasia.

<table>
<thead>
<tr>
<th></th>
<th>$E_L$ (MPa)</th>
<th>$E_R$ (MPa)</th>
<th>$E_T$ (MPa)</th>
<th>$\mu_{LR}$</th>
<th>$\mu_{LT}$</th>
<th>$\mu_{RT}$</th>
<th>$G_{LT}$ (MPa)</th>
<th>$G_{LR}$ (MPa)</th>
<th>$G_{RT}$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bangkirai</td>
<td>13457</td>
<td>2072</td>
<td>1103</td>
<td>0.35</td>
<td>0.45</td>
<td>0.56</td>
<td>1090</td>
<td>1198</td>
<td>283</td>
</tr>
<tr>
<td>Meranti</td>
<td>5529</td>
<td>851</td>
<td>453</td>
<td>0.35</td>
<td>0.45</td>
<td>0.56</td>
<td>448</td>
<td>492</td>
<td>116</td>
</tr>
<tr>
<td>Albasia</td>
<td>3400</td>
<td>524</td>
<td>279</td>
<td>0.35</td>
<td>0.45</td>
<td>0.56</td>
<td>275</td>
<td>303</td>
<td>71</td>
</tr>
</tbody>
</table>

4. Discussion

As shown in the analyses of beams subject to various loading conditions using FEM, orthotropic material always results in a lower critical load than isotropic material. The difference between the two materials becomes larger if the beam has cross grain. While investigating various angles of cross grain using real beam specimen is difficult, FEM is a versatile tool to perform such task.
Investigation on various depth/width ratio of the beam cross section shows that as the ratio increases, the difference between isotropic and orthotropic materials become larger.

5. Conclusions

Since orthotropic assumption always results in lower critical load for the loading cases considered in this paper, then isotropic equations for LTB should not be used for orthotropic material such as wood. One possible approach is to use reduction factors in the isotropic equations. Such factor should depend on beam dimensions, specific gravity, cross grain, and perhaps all other possible factors that have not been considered in this paper.

6. Acknowledgements

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7. References


