Three Dimensional Stability Analysis of Wood Beam-columns

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1. Summary

A three dimensional stability analysis procedure is presented for the determination of the ultimate strength of wood beam column subject to axial compression load and biaxial bending moments. The analysis takes into account the material nonlinearity, geometrical changes and variation of wood mechanical properties. The procedure is applicable to wood beam columns with various boundary conditions and lateral bracings. Comparisons with a commercial Finite Element program ANSYS and formulas from Structural Stability Research Council (SSRC) indicate the proposed analysis procedure is fast, robust, and accurate.

2. Introduction

Wood roof truss systems are widely used in residential and lightframe low-rise commercial construction in North America. Most wood members used in these truss systems are considered slender as they are sized with 2-by series 38mm thickness cross section. Therefore, stability analysis is very important to safeguard against lateral buckling failure of the compression members.

This area of study, however, has not been fully developed. Existing methods for stability analysis and lateral bracing design are mostly based on linear elastic material model (Tsien 1942, Throop 1947, Winter 1958, Medland 1977, Plaut and Yang 1993, Plaut 1993, Munch-Andersen 2004). Some efforts have been made to take into account the material nonlinearity and geometry changes by using the Column Deflection Curve method (CDC) or Finite Element Method (FEM) (Buchanan 1984, Koka 1987, Lau 2000). However, none of these models considers the three dimensional behavior of compression wood columns subject to biaxial bending moments.

The present study aims to develop a nonlinear three dimensional column stability analysis procedure by assessing the load carrying capacity of wood beam column under axial compression load and biaxial bending moments. The proposed approach is based on three dimensional CDC method and can be applied to a variety of boundary conditions with lateral bracing provisions. Material nonlinearity, P-Delta effect and size effect of wood mechanical properties are considered. Numerical analysis results are verified against predictions from commercial finite element program ANSYS and formulas from SSRC.

3. Numerical Analysis

3.1. Basic Assumptions:

1) The numerical analysis is based upon the following assumptions:
2) Plane section remains plane after deformation
3) Shear stress is uniformly distributed over the cross section
4) Stress strain relationships are known and independent of rate of loading
5) Deflection is small compared to the size of the members
6) Torsional stress and twist deformation are negligible

3.2. Method of Analysis

The numerical analysis uses the CDC method to construct the deflection curve of the beam column members subject to axial compression load and biaxial bending moments. The ultimate load carrying capacity of the beam column member can be obtained by trial and error process in which the external loads are increased until instability. The pertinent information required to run CDC method consists of the moment-curvature-axial load relation and the boundary conditions of the structural member.

3.3. Generation of moment-curvature-axial load relation

The moment-curvature-load relation can be used to get the curvatures given the external axial load and biaxial bending moments. The deformation of a cross section subject to biaxial bending moments and axial load can be represented by the curvatures in the direction of two principle axes of the cross section, \( \phi_x, \phi_y \), and the uniform strain \( \varepsilon_0 \) as shown in Fig 1.

![Fig 1 Cross section deformations under biaxial bending and axial load](image)

The entire cross section is divided into \( n \) rectangular elements. The strain and stress distribution over each element are assumed uniform, therefore, they can be represented by the corresponding terms at the centroid of the element \((x_i, y_i)\).

Given an element with the centroid coordinates \((x_i, y_i)\), the normal strain \( \varepsilon_i \) of the element can be determined from the curvatures and uniform strain of the cross section as

\[
\varepsilon_i = y_i \phi_x + x_i \phi_y + \varepsilon_0
\]

The normal stress can be obtained in terms of normal strain

\[
\sigma_i = E_i \varepsilon_i
\]

where \( E_i \) is the secant modulus of elasticity from the prescribed stress strain relation.

The stress resultant of the cross section \( P, M_x \) and \( M_y \) can be obtained from section equilibrium equations:

\[
M_x = \sum_{i=1}^{n} \sigma_i A_i y_i = \sum_{i=1}^{n} E_i (\varepsilon_0 + y \phi_x + x \phi_y) A_i y_i
\]
\[ M_y = \sum_{i=1}^{n} \sigma_i A_i x_i = \sum_{i=1}^{n} E_i (\varepsilon_0 + y \phi_x + x \phi_y) A_i x_i \]  

\[ P = \sum_{i=1}^{n} \sigma_i A_i = \sum_{i=1}^{n} E_i (\varepsilon_0 + y \phi_x + x \phi_y) A_i \]

where \( A_i \) is the area of each element of the cross section.

The moment-curvature-load diagram, which defines the relationships between the stress resultants and the internal deformations \( \varepsilon_0, \phi_x \) and \( \phi_y \), can be constructed by repeatedly solving \( \varepsilon_0, \phi_x \) and \( \phi_y \) from equations [3] to [5] for given \( M_{ext,x}, M_{ext,y} \) and \( P \).

Due to the nonlinearity present in the material stress strain relation, an iterative searching scheme is employed to search for the internal deformations \( \varepsilon_0, \phi_x \) and \( \phi_y \) that will produce the prescribed \( M_{ext,x}, M_{ext,y} \) and \( P \).

### 3.4. Construction of column deflection curve by CDC method

CDC method is a piecewise numerical integration method for plastic beam column members. It treats the beam column as a series of discrete segments and calculates the curvatures at the division points according to the axial load and bending moments. The curvatures are then integrated throughout the length of the segment to obtain the deflections of each division joint. The numerical process of the CDC method is explained with the aid of Fig 2.

![Fig 2 Column Deflection Curve analysis model](image)

The beam column member is divided into \( n \) segments, each segment starts at division joint \( i \) and ends at division joint \( i+1 \). The deflection of each division joint consists of the initial deflection \( e_{i,x,0} \) and \( e_{i,y,0} \), and the secondary deflection caused by external loads, \( e_{i,x} \) and \( e_{i,y} \) in \( x \) and \( y \) axis, respectively.

For division joint \( i \), the internal biaxial bending moments \( M_{x,i} \) and \( M_{y,i} \) can be calculated by
where $N$ denotes the axial load, $M_{x,i,k}$ and $M_{y,i,k}$ are the restoring bending moments due to the lateral bracings. The lateral bracings are characterized by the axial stiffness $k_x$ and $k_y$.

If the lateral bracings are located at mid-span of the beam-column, $M_{x,i,k}$ and $M_{y,i,k}$ at joint $i$ that is spaced from the bottom joint by $z_i$ can be calculated by

\begin{align}
M_{x,i}(k_x) &= \begin{cases} 
0.5k_x e_{mid,y} z_i & 0 \leq z_i \leq L/2 \\
0.5k_x e_{mid,y} (L - z_i) & L/2 \leq z_i \leq L
\end{cases} \\
M_{y,i}(k_x) &= \begin{cases} 
0.5k_y e_{mid,x} z_i & 0 \leq z_i \leq L/2 \\
0.5k_y e_{mid,x} (L - z_i) & L/2 \leq z_i \leq L
\end{cases}
\end{align}

where, $e_{mid,x}$ and $e_{mid,y}$ are the deflections at the mid-span joint.

For given external biaxial bending moments and axial load, the curvatures of the cross section $\phi_{x,i}$ and $\phi_{y,i}$ can be obtained by using the searching scheme discussed in previous section. Then the deflections at division $i + 1$ respect to joint $i$ can be approximated by Taylor extension series as

\begin{align}
\delta_{i+1,x} &= \theta_{x,i} l + \frac{1}{2} \phi_{x,i} l^2 + \frac{1}{6} \nu_{x,i} l^3 \\
\delta_{i+1,y} &= \theta_{y,i} l + \frac{1}{2} \phi_{y,i} l^2 + \frac{1}{6} \nu_{y,i} l^3
\end{align}

where $l$ is the length of the column segment, $\theta_{x,i}$ and $\theta_{y,i}$ are the angles of the rotation at joint $i + 1$, $\phi_{x,i}$ and $\nu_{x,i}$ are the rate of change of the curvatures.

Eventually, the total deflections at joint $i + 1$ can be calculated as

\begin{align}
e_{i+1,x} &= e_{i,x} + \delta_{i+1,x} \\
e_{i+1,y} &= e_{i,y} + \delta_{i+1,y}
\end{align}

Same procedures will run from the bottom joint to the top joint. Some initial trial values will be assigned to the bottom joint and the calculated deformations at top joint are used to check with the prescribed boundary conditions. Trial and error process will be employed for the initial trial values until the convergence criterion is satisfied.

### 3.5 Convergence criterion and calculation of lateral bracings

The convergence criterion of CDC method and the determination of the initial trial values can be established from the boundary conditions of the structural member. Generally, the restraint terms of the essential boundary conditions, such as rotation for fix-end support, can be used as convergence criterion. On the other hand, the unrestraint terms, such as the rotation of a roller support, can be used as the initial trial values. The CDC iteration method is considered converged when the error of the displacements at the top joint is within preset tolerance. The divergence of the CDC iteration method is considered as the indication of the instability failure.
Calculations involving lateral bracings also use the trial and error process. At the beginning of each iteration, presumed deflection values are assigned to the lateral bracings, based on which, the lateral bracing forces are calculated. Then the deflection curve of the beam column will be constructed for certain external loads and the lateral forces. The obtained lateral bracings deflections will be checked against the presumed values and the nonlinear iterative procedure will be repeated until the errors are within preset tolerance.

3.6. Size effect of wood mechanical properties

Determination of size effect of wood mechanical properties requires information about the dimension and loading conditions of the structural member. The length, width and depth effects could be quantified given the dimensions of the beam column segments. The loading condition effect is coupled with the stress distribution over the cross section and therefore has to be determined by iteration.

The size effect factor of strength values of uniformly stressed wood member could be expressed as

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1}\right)^{\frac{1}{k}} \quad [10]$$

where $\sigma_1$ and $\sigma_2$ are the strength of two specimen of different volume $V_1$ and $V_2$, $k$ is the shape parameter of Weibull distribution of wood strength and can be derived from the experiment data.

4. Program Validation

4.1. Midspan deflection compared with ANSYS results

Because of the lack of test data on biaxial loaded wood beam column members, the program was validated by comparing results with commercial Finite Element program ANSYS prediction. A wood beam column member that is subject to axial compression load and biaxial bending moments was analyzed and the deflections of the mid-span joint were compared.

The wood member was made from 38×89 (nominal 2 by 4) Spruce-Pine-Fir, No.2 grade, 1000 mm long. According to CWC lumber properties project (Barrett, 1994), the mean value of Modulus of Elasticity (MOE) is 9915 MPa, compressive strength (UCS) is 28.18 MPa, tension strength (UTS) is 23.27 MPa. Length effect was applied to the strength values of the wood member of different size from standard specimen in CWC lumber properties project with $k$ factor as 13.0 in compression and 5.0 in tension (Buchanan, 1987). The constitutive relation was assumed to be elastoplastic in compression and linear in tension.

The initial deflection of the beam column member was assumed to be of sinusoid shape, with the maximum value at the midspan to be 5 mm. The end eccentricity of the load is 1 mm. Both the initial deflection and end eccentricity of axial load were assumed to have an angle of 30 degree with the weak axis ($x$) of the cross section. The lateral bracings were located at the midspan, with same cross sectional dimension and material properties, 500 mm long, in both principle axes of the cross section of the structural member, respectively.

Two types of boundary conditions were considered, one is simply supported, and the other is fix-end supported at one end and roller supported at the other end, as shown in Fig 3.
Fig 3 Structural forms of wood beam column member

Results are listed in Table 1, from which, it can be seen that the deflections of midspan joint obtained from the proposed program agree quite well with ANSYS prediction.

<table>
<thead>
<tr>
<th>Axial load (N)</th>
<th>Load case (a)</th>
<th>Load case (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PP (10^-3 mm)</td>
<td>ANSYS (10^-3 mm)</td>
</tr>
<tr>
<td>10% $P_u$ *</td>
<td>3.97 2.26</td>
<td>3.88 2.30</td>
</tr>
<tr>
<td>15% $P_u$</td>
<td>5.98 3.39</td>
<td>5.84 3.47</td>
</tr>
<tr>
<td>20% $P_u$</td>
<td>8.00 4.52</td>
<td>7.80 4.64</td>
</tr>
<tr>
<td>25% $P_u$</td>
<td>10.03 5.66</td>
<td>9.75 5.82</td>
</tr>
<tr>
<td>30% $P_u$</td>
<td>12.08 6.79</td>
<td>11.71 7.01</td>
</tr>
</tbody>
</table>

* $P_u$ is the compression capacity of the column with no consideration of buckling failure, equals 102,060N in this case

4.2. Ultimate strength of beam-column under uniaxial compression load

In this case, the ultimate strength of a steel beam-column was studied. The ultimate strength of the beam-column member with various slenderness ratios was calculated by the proposed program and compared with the suggested formula from The Structural Stability Research Council (SSRC) (Salimon and Johnson, 1990). The simply support steel member was considered with a cross section of 38 x 89 mm as an example. Lateral bracing was not considered.

The stress strain relation of steel was assumed to be elastoplastic, with modulus of elasticity $E_0$ as 2.06E+5 MPa, and yield stress $f_y$ as 400 MPa. The column strength and slenderness ratio curves from SSRC and proposed approach are plotted in Fig 4. For rectangular cross section, the slenderness ratio was calculated based on the performance of the weak axis of the cross section. Therefore, the slenderness ratio was calculated by $\lambda = \frac{L}{w}$, where $L$ is the efficient length of the column, $w$ is the width of the cross section.

![Fig 4 Ultimate strength of beam-column of various lengths](image-url)
From Fig 4, it can be seen the results from proposed approach fall between the curves from SSRC and the overall agreement is good.

5. Conclusion

A three dimensional stability analysis program is presented to calculate the load carrying capacity of wood beam column member subject to compression load and biaxial moments and study the lateral bracing requirements. The proposed program takes into account the material nonlinearity, geometrical changes and variation of wood mechanical properties. Validation with ANSYS program and formulas from SSRC were made and good agreement was achieved.

6. References